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On Approximate Minima of a Convex Functional and Lower Semicontinuity of Metric Projections

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In this paper we are concerned with the continuity of the set-valued mapping whose values are approximate minima of constrained minimization problems when the constraint sets are perturbed in a reflexive Banach space and the space of constraint sets is equipped with the topology of Mosco convergence. This leads to an interesting generic theorem on points of single-valuedness of minima of a given function f on nonempty closed convex subsets of a reflexive Banach space. Using the continuity results of the general framework, a characterization of lower semicontinuity of metric projections is given.

1. INTRODUCTION

For many practical and theoretical reasons it is often of interest to study the stability of solutions of optimization problems under various perturbations of a given problem. Typically, the perturbations may be due to the rounding errors in numerical computations, approximations used in solving a given problem, or perhaps due to variations of parameters involved. Continuous dependence of the optimal values and of the sets of optimal solutions on perturbations has been explored by several authors (cf., e.g., [8, 10–12, 14–16, 21, 29]). An alternative approach for studying the stability question, which seems more tractable from a numerical point of view, relies on the concept of ε-approximate solutions. Recently, this concept has been utilized for both qualitative as well as quantitative investigation of stability in optimization [15, 29, 2, 11].

In the framework of a reflexive Banach space X, apparently the most useful notion of convergence for a sequence of nonempty closed convex subsets $CC(X)$ of X is the so-called Mosco convergence introduced by U. Mosco in [18]. Specifically, a sequence $\langle C_n \rangle$ in $CC(X)$ is said to be *Mosco convergent* to an element C in $CC(X)$, written $C_n \to^M C$, if (M_1) for each x in C there exists a sequence $\langle x_n \rangle$ convergent to x such that for each

0021-9045/91 $3.00

n, $x_n \in C_n$, and (M_2) whenever $\langle n(i) \rangle$ is an increasing sequence of positive integers and $x_{n(i)} \in C_{n(i)}$ for each i, then the weak convergence of $\langle x_{n(i)} \rangle$ to $x \in X$ implies $x \in C$. Identifying lower semicontinuous proper convex functions $\Gamma_0(X)$ with their epigraphs, Mosco convergence of a sequence $\langle f_n \rangle$ in $\Gamma_0(X)$ is defined as the Mosco convergence of their epigraphs. Equivalently, the sequence $\langle f_n \rangle$ is said to be Mosco convergent to an element f if (M_1') for each x in X, there is a sequence $\langle x_n \rangle$ convergent to x such that $f(x) = \lim_n f_n(x_n)$ and (M_2') whenever a sequence $\langle x_n \rangle$ in X is weakly convergent to an element x in X, then $f(x) \leqslant \underline{\lim}_n f_n(x_n)$.

For a variety of applications of Mosco convergence, one may consult the comprehensive monograph [1] and, in particular, [4, 24] for approximation theory, [28] for control theory, and [18, 22] for variational inequalities.

In spite of great interest [1, 22] it is only recently in [3] that a "hit-and-miss" topology τ_M of the Vietoris type was introduced on $CC(X)$, for X a Banach space, which is compatible with Mosco convergence in $CC(X)$. In terms of the standard plus and minus notation for hyperspaces, the *Mosco topology* τ_M is generated by all sets of the form

$$V^- \equiv \{A \in CC(X) : A \cap V \neq \emptyset\}$$

$$(K^c)^+ \equiv \{A \in CC(X) : A \cap K = \emptyset\},$$

where V is an open subset of X and K is a weakly compact subset of X. For a detailed investigation of this topology in relation to the other hyperspace topologies of interest we refer the reader to [3].

This paper is in two parts. The first part synthesizes continuity considerations of ε-approximate solutions for minimization problems. The second part deals with applications to metric projections. Section 3 treats continuity of the marginal function and of the set-valued function whose values are ε-approximate solutions of constrained minimization problems when the constraint sets are perturbed in a reflexive Banach space X and the space $CC(X)$ of constraint sets is equipped with the Mosco topology. Section 4 deals with upper semicontinuity of the ε-approximate solutions and of the optimal solutions. This leads to an interesting generic theorem for single-valuedness of points of minima of a given function f on $CC(X)$. Employing the continuity results in the general framework of Section 3, a characterization of lower semicontinuity of metric projection is discussed in Section 5.

2. Preliminaries

In the sequel, X will be a normed linear space and w will denote the weak topology of X. The origin and closed unit ball of X will be denoted by θ and U, respectively. $B_\varepsilon(x_0)$ will denote the open ball of center x_0 and radius ε. Also S (resp. S^*) will denote the unit sphere (norm one elements) of X (resp. the normed dual of X). We write $d(x; C)$ for the distance between x and the set C, and denote by $P(x; C)$ the set $\{y \in C: \|x-y\| = d(x; C)\}$. We denote by (R_f) the class of reflexive Banach spaces, by (R) the class of rotund (strictly convex) normed spaces, and by (R^*) the class of normed spaces whose duals are in (R). Likewise, we denote by (H) (resp. (H^*)) the class of normed spaces for which weak (resp. weak*) convergence of a net in S (resp. S^*) implies its norm convergence. Following Brown [6], we denote by (P) the class of normed linear spaces in which for every pair of elements x, y in X with $\|x+y\| \leqslant \|x\|$, there exist positive constants ε, δ such that $\|z+\varepsilon y\| \leqslant \|z\|$ whenever $\|x-z\| \leqslant \delta$. In addition to the class $CC(X)$ of nonempty closed convex subsets of X employed in the Introduction, we denote by $CL(X)$ (resp. $WK(X)$) the class of nonempty closed (resp. nonempty weakly compact) subsets of X.

If T, Y are topological spaces, we call a multifunction $\Gamma: T \rightrightarrows Y$ (by which we mean a set-valued function from T to $CL(Y)$) lower semicontinuous, abbreviated l.s.c. (resp. upper semicontinuous abbreviated u.s.c.) if for each open subset V of Y the set $\{t \in T: \Gamma(t) \in V^-\}$ (resp. the set $\{t \in T: \Gamma(t) \in V^+\}$) is open in T. Γ is said to be continuous if it is both l.s.c. as well as u.s.c. If Γ is u.s.c. and with compact values, then Γ is called an *usco* map [7]. If X is equipped with the topology w, then we employ the term w-u.s.c. (resp. w-usco) for u.s.c. (resp. usco) map into X so topologized.

Given a function $f: X \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, we denote by $\text{sub}(f; \alpha)$ the sublevel set $\{x \in X: f(x) \leqslant \alpha\}$ of f at height α. The function f is said to be *inf-bounded* (resp. *w-inf-compact*) if $\text{sub}(f; \alpha)$ is bounded (resp. w-compact) for each $\alpha \in \mathbb{R}$. In case f is convex, the conditions ensuring inf-boundedness, w-compactness of f are given, e.g., in [20, Theorem 10]. (Compare also [16, Lemma 2.5] for an alternative condition for inf-boundedness.)

In the sequel $\Sigma(X)$ (resp. $\Lambda(X)$) will denote the class of real functions on X which are continuous and w-inf-compact (resp. convex, continuous, and inf-bounded). Clearly if $X \in (R_f)$, then $\Lambda(X) \subset \Sigma(X)$. It is also easily observed that if $K \in WK(X)$, then $x \to d(x; K)$ is w.l.s.c. This follows immediately from the w-lower semicontinuity of the norm. Since in this case the function $d(\cdot; K)$ is continuous and inf-bounded, we note that it is, in fact, in $\Sigma(X)$ provided $X \in (R_f)$.

3. Stability of ε-Approximate Solutions

For most of this section, we assume X to be in (R_f). Given a function $f: X \to \mathbb{R}$, a set C in $CC(X)$, and a number $\varepsilon > 0$ we denote by $v_f(C)$ the number $\inf f(C)$ and by $\arg\min_f(C)$ (resp. ε-$\arg\min_f(C)$) the set $\{x \in C: f(x) = v(C)\}$ (resp. the set $\{x \in C: f(x) \leqslant v_f(C) + \varepsilon\}$). Note that without additional hypothesis $\arg\min_f(C)$ may be empty; but ε-$\arg\min_f(C)$ is nonempty by definition, provided $v_f(C) > -\infty$. Our first result gives a characterization of the topology τ_M on $CC(X)$ in terms of continuity of the marginal functions $C \to v_f(C)$ for $f \in \Sigma(X)$.

Theorem 3.1. *Suppose $X \in (R_f)$. Then τ_M is the weakest topology on $CC(X)$ for which $v_f: \langle CC(X), \tau_M \rangle \to \mathbb{R}$ is continuous for each $f \in \Sigma(X)$.*

Proof. We first show that if $f \in \Sigma(X)$ then v_f is continuous. Fix $C_0 \in CC(X)$ and suppose $v_f(C_0) < \alpha$ for some $\alpha \in \mathbb{R}$. Pick up $x_0 \in C_0$ such that $f(x_0) < \alpha$. Since f is u.s.c., there is an open neighborhood V of x_0 such that $f(x) < \alpha$ for all $x \in V$. Clearly $C_0 \in V^-$. For $C \in V^-$ and $x \in C \cap V$ we have $v_f(C) \leqslant f(x) < \alpha$, which proves that v_f is u.s.c. at $C = C_0$. Lower semicontinuity of the functional at C_0 holds trivially if $v_f(C_0) = v_f(X)$. Suppose $v_f(C_0) \neq v_f(X)$ and let $v_f(C_0) > \alpha$ for some $\alpha \in \mathbb{R}$. Let β be an arbitrary number such that $\beta \geqslant v_f(X)$ and $v_f(C_0) > \beta > \alpha$. Let $K = \text{sub}(f; \beta)$. Then $K \in WK(X)$. Obviously $C_0 \in (K^c)^+$ and for $C \in (K^c)^+$ we have $f(x) > \beta$ for $x \in C$. Therefore $v_f(C) \geqslant \beta > \alpha$, which proves that v_f is l.s.c. at $C = C_0$.

It remains to show that if v_f is τ-continuous for a topology τ on $CC(X)$ for each $f \in \Sigma(X)$, then $\tau_M \subset \tau$. To this end, let $\langle C_\lambda \rangle$ be a net in $CC(X)$ such that τ-$\lim_\lambda C_\lambda = C_0 \in CC(X)$. It suffices to show that τ_M-$\lim_\lambda C_\lambda = C_0$. First, suppose $C_0 \in V^-$ for some open subset V of X. Pick $x_0 \in C_0$ and $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset V$. Let $f(x) = \|x_0 - x\|$ for $x \in X$. Then $f \in \Sigma(X)$ and $v_f(C_0) = d(x_0; C_0) = 0$. Since by hypothesis

$$\lim_\lambda v_f(C_\lambda) = \lim_\lambda d(x_0; C_\lambda) = v_f(C_0) = 0,$$

we have $C_\lambda \subset B_\varepsilon(x_0)^- \subset V^-$ eventually. Now let $C_0 \in (K^c)^+$ for some $K \in WK(X)$ and assume in order to get a contradiction that $C_\mu \cap K \neq \varnothing$ for some subnet $\langle C_\mu \rangle$ of the net $\langle C_\lambda \rangle$. Let K_0 be the w-closure of the set $(\bigcup_\mu C_\mu) \cap K$ and let $f(x) = d(x, K_0)$, $x \in X$. Then $f \in \Sigma(X)$ and $v_f(C_\mu) = 0$ for each μ, but $v_f(C_0) = \inf_{x \in C_0} d(x; K_0) > 0$, which is a contradiction. Therefore $C_\lambda \in (K^c)^+$ eventually and this proves that τ_M-$\lim_\lambda C_\lambda = C_0$. ∎

Remark 3.2. We note that the first part of the preceding theorem, v_f is τ_M-continuous on $CC(X)$, holds for an arbitrary normed linear space.

We now come to our main result of this section.

THEOREM 3.3. *If* $X \in (R_f)$ *and* $f \in \Lambda(X)$, *then the map* $\varepsilon\text{-arg min}_f$: $\langle CC(X), \tau_M \rangle \to \langle CC(X), \tau_M \rangle$ *is continuous.*

Proof. Clearly $\varepsilon\text{-arg min}_f(C)$ is in $CC(X)$ for each $C \in CC(X)$. Let $\langle C_\lambda \rangle_{\lambda \in \Lambda}$ be a net in $CC(X)$ which is τ_M-convergent to C_0 in $CC(X)$. To prove the desired continuity it suffices to show that

(i) if $\varepsilon\text{-arg min}_f(C_0) \in V^-$ for an open set V, then $\varepsilon\text{-arg min}_f(C_\lambda) \in V^-$ eventually, and

(ii) if $\varepsilon\text{-arg min}_f(C_0) \in (K^c)^+$ for a set $K \in WK(X)$, then $\varepsilon\text{-arg min}_f(C_\lambda) \in (K^c)^+$ eventually.

To prove (i), suppose $\varepsilon\text{-arg min}_f(C_0) \cap V \neq \varnothing$. By convexity of f, there is a point $x_0 \in \varepsilon\text{-arg min}_f(C_0) \cap V$ such that $f(x_0) < v_f(C_0) + \varepsilon$. Using continuity of f and Theorem 3.1, there is a τ_M-open set $\mathfrak{N}$ in $CC(X)$ containing C_0 and an open set V_1 containing x_0 such that whenever $x \in V_1$ and $C \in \mathfrak{N}$, then

$$f(x) < v_f(C) + \varepsilon. \tag{*}$$

We may assume, without loss of generality, that $V_1 \subset V$. Since $C_0 \in V_1^- \cap \mathfrak{N}$ and the net $\langle C_\lambda \rangle$ is τ_M-convergent to C_0, there is an index μ such that $C_\lambda \in V_1^- \cap \mathfrak{N}$ for $\lambda \geqslant \mu$. Therefore for each $\lambda \geqslant \mu$, $C_\lambda \in \mathfrak{N}$ and there is a point $x_\lambda \in C_\lambda \cap V_1$. By (*) we get $f(x_\lambda) < v_f(C_\lambda) + \varepsilon$, which shows that $\varepsilon\text{-arg min}_f(C_\lambda) \cap V \neq \varnothing$, that is, $\varepsilon\text{-arg min}_f(C_\lambda) \in V^-$ for $\lambda \geqslant \mu$.

To prove (ii), it suffices to show that if K is a weakly compact subset of X such that $\varepsilon\text{-arg min}_f(C_\mu) \cap K \neq \varnothing$ for a subnet $\langle C_\mu \rangle$ of the net $\langle C_\lambda \rangle$, then $\varepsilon\text{-arg min}_f(C_0) \cap K \neq \varnothing$. Indeed, let $x_\mu \in \varepsilon\text{-arg min}_f(C_\mu) \cap K$. Then $\langle x_\mu \rangle$ has a subnset $\langle x_\nu \rangle$ w-convergent to an element $x_0 \in K$. Since $\langle x_\nu \rangle$ is bounded, by Lemma 3.5 of [4] we have $x_0 \in C_0$. Also since $X \in (R_f)$ and $f \in \Sigma(X)$, using Theorem 3.1, we have

$$f(x_0) \leqslant \varliminf_\nu f(x_\nu) \leqslant \varliminf_\nu v_f(C_\nu) + \varepsilon \leqslant v_f(C_0) + \varepsilon.$$

Therefore $x_0 \in \varepsilon\text{-arg min}_f(C_0) \cap K$ and this completes the proof of (ii). ∎

Recall (cf., e.g., [4]) that given a bounded subset F and a nonempty subset C of X, the Chebyshev radius $\text{rad}(F; C)$ of F in C is the number $\inf\{r(F; x) : x \in C\}$ where $r(F; x) \equiv \sup\{\|x - y\| : y \in F\}$. Any point $x \in C$ for which $r(F; x) = \text{rad}(F; C)$ is called a relative Chebyshev center of F in C and the (possibly empty) set of relative Chebyshev centers of F in C is denoted by $\text{Cent}(F; C)$. Evidently, since $\text{sub}(r(F; \cdot); \alpha) \subset (\alpha + \text{diam}(F))U + F$, the convex continuous function $x \to r(F; x)$ is inf-bounded. Theorem 3.3 is, therefore, applicable to the approximate relative chebyshev center map $\varepsilon\text{-Cent}(F; \cdot) : CC(X) \to CC(X)$ where

$$\varepsilon\text{-Cent}(F; C) \equiv \{x \in C : r(F; x) \leqslant \text{rad}(F; C) + \varepsilon\}$$

as well as to its particular case: the approximate metric projection map $\varepsilon - P(x; \cdot): CC(X) \to CC(X)$ where

$$\varepsilon - P(x; C) \equiv \{ y \in C : \|x - y\| \leqslant d(x; C) + \varepsilon \}.$$

This yields

THEOREM 3.4. *Suppose $X \in (R_f)$ and C_n, C_0 are in $CC(X)$. Consider the following statements.*

(1) $C_n \to^M C_0$;

(2) *for every* $\varepsilon > 0$, $\varepsilon\text{-Cent}(F; C_n) \to^M \varepsilon\text{-Cent}(F; C_0)$ *for each bounded subset F of X*;

(3) *for every* $\varepsilon > 0$, $\varepsilon - P(x; C_n) \to^M \varepsilon - P(x; C_0)$ *for each x in X*;

(4) *for every* $\varepsilon > 0$, $d(\cdot; \varepsilon - P(x; C_n)) \to^M (d(\cdot; \varepsilon - P(x; C_0))$ *for each x in X*;

(5) $d(\cdot; P(x; C_n)) \to^M d(\cdot; P(x; C_0))$ *for each x in X*;

(6) $P(x; C_n) \to^M P(x; C_0)$ *for each x in X*;

(7) $d(x; C_n) \to d(x; C_0)$ *for each x in X*;

(8) $\text{rad}(F; C_n) \to \text{rad}(F; C_0)$ *for every bounded subset F of X admitting farthest point*;

(9) $\text{Cent}(F; C_n) \to^M \text{Cent}(F; C_0)$ *for every bounded subset F of X admitting farthest point.*

We have $(1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (4)$; $(5) \Leftrightarrow (6) \Rightarrow (7)$ *and* $(1) \Rightarrow (8) \Rightarrow (7)$. *If $X \in (R) \cap (H)$ than* $(4) \Rightarrow (5)$ *and* $(1) \Rightarrow (9)$. *Furthermore, if $X \in (H^*)$ then* $(7) \Rightarrow (1)$. *Thus for X in $(Rf) \cap (R) \cap (H) \cap (H^*)$ all the nine statements are equivalent.*

Proof. $(2) \Rightarrow (2) \Rightarrow (3)$. These implications follow immediately from Theorem 3.3 and the preceding observations. The implication $(1) \Rightarrow (3)$ is observed in [19, Proposition 2.3] with a different proof.

$(3) \Leftrightarrow (4)$. These follow immediately from Theorem 2.1 of [22].

$(6) \Rightarrow (7)$. This is easy to see from Proposition 2.2 of [22].

$(1) \Rightarrow (8)$. This follows from Theorem 3.1.

$(4) \Rightarrow (5)$. Under the hypothesis, $X \in (R) \cap (H)$. Since $X \in (R)$, let $P(x; C_n) = \{z_n\}$ a singleton for each n and $P(x; C_0) = \{z_0\}$. Since Mosco convergence of a sequence of distance functions implies its pointwise convergence [22, p. II.4], we have

$$\|x - z_n\| = d(x; \varepsilon - P(x; C_n)) \to d(x; \varepsilon - P(x; C_0)) = \|x - z_0\|.$$

Since $X \in (R_f)$, $\langle z_n \rangle$ has a subsequence $\langle z_{n(k)} \rangle$ w-convergent to an element w_0; by (3) we have $w_0 \in \varepsilon - P(x; C_0)$ for every $\varepsilon > 0$. Since $P(x; C_0) = \bigcap_{\varepsilon > 0} \varepsilon - P(x; C_0) = \{z_0\}$, we have $w_0 = z_0$. Since $X \in (H)$, we obtain

$z_{n(k)} \to z_0$. Since every subsequence of $\langle z_n \rangle$ has a subsequence converging to z_0, we conclude that $z_n \to z_0$ and (5) follows immediately from the weak lower semicontinuity of the norm function.

(1) ⇒ (9). Under the assumption $X \in (R) \cap (H)$. This is shown in [4, Cor. 3.11].

(7) ⇒ (1). Under the assumption $X \in (H^*)$. This is proved in [5, Thm. 3.4]. ∎

4. Baire Category Results

We first consider the upper semicontinuity of the multifunctions ε-arg $\min_f$ and arg $\min_f$ which lead us to explore a generic theorem on points of single valuedness of the multifunction arg $\min_f$ in this section. Throughout this section X will be in (R_f).

Theorem 4.1. *Let $f \in \Lambda(X)$, then for each $\varepsilon > 0$ the multifunction ε-arg $\min_f$: $(CC(X), \tau_M) \to CC(X)$ is w-usco.*

Proof. The w-compactness of ε-arg $\min_f(C)$ for $C \in CC(X)$ follows immediately. To prove that ε-arg $\min_f$ is w-usco, it suffices, therefore, to show that

$$\varepsilon\text{-arg}\min{}_f^{-1}(E) \equiv \{C \in CC(X) \colon \varepsilon\text{-arg}\min{}_f(C) \cap E \neq \varnothing\}$$

is τ_M-closed for each w-closed subset E of X. To this end, let $\langle C_\lambda \rangle$ be a net in ε-arg $\min_f^{-1}(E)$ which is τ_M-convergent to C_0. For each index λ, choose a point x_λ in ε-arg $\min_f(C_\lambda) \cap E$. By Theorem 3.1, $v_f(C_\lambda) \to v_f(C_0)$, and since f is w-inf-compact the net $\langle x_\lambda \rangle$ has a subnet $\langle x_\mu \rangle$ w-convergent to an element $x_0 \in E$. By Lemma 3.3 of [4], $x_0 \in C_0$ and by the same argument as in part (ii) of Theorem 3.3 we conclude that $x_0 \in \varepsilon$-arg $\min_f(C_0) \cap E$. This completes the proof of ε-arg $\min_f$ being w-usco. ∎

We observe that the proof of the preceding theorem also applies in case $\varepsilon = 0$. This yields

Corollary 4.2. *Let $f \in \Lambda(X)$, then the multifunction* arg $\min_f$: *$(CC(X), \tau_M) \to CC(X)$ is w-usco.*

By Theorem 4.3 of [3], when X is in (R_f) and separable, $CC(X)$ equipped with Mosco topology τ_M is a Polish space (second countable and completely metrizable). Since $(CC(X), \tau_M)$ is a Baire space, it appears meaningful to ask: For a given $f \in \Lambda(X)$, is arg $\min_f(C)$ single-valued for most C in $CC(X)$ in the sense of Baire category? The following example shows that the answer is negative. In fact the collection of sets C for which arg $\min_f(C)$ is single-valued need not even be dense in $CC(X)$.

EXAMPLE 4.3. Let $X = \mathbb{R}^2$, equipped with the Euclidean norm. For each point (α_1, α_2) in $\mathbb{R}^2$, let

$$V(\alpha_1, \alpha_2) := \{(\beta_1, \beta_2) : \|(\beta_1, \beta_2) - (\alpha_1, \alpha_2)\| \leqslant 0.1\}.$$

Let $f((x_1, x_2)) = d((x_1, x_2); V(0, 0))$, $(x_1, x_2) \in \mathbb{R}^2$. Clearly $f \in \Lambda(X)$. Let

$$\mathfrak{N} \equiv V(1, 1)^- \cap V(1, -1)^- \cap V(-1, 1)^- \cap V(-1, -1)^-.$$

Then $\mathfrak{N}$ is τ_M-open in $CC(X)$ and for C in $\mathfrak{N}$, we have $\arg\min_f(C) = V(0, 0)$.

In the preceding example, the constraint sets C are allowed to intersect $\arg\min_f(X)$. We note that in this case $\operatorname{int} \arg\min_f(X) \neq \varnothing$ and the constraint sets C, are in fact, allowed to intersect $\operatorname{int} \arg\min_f(X)$. A simple modification of the preceding example shows that the answer to the above mentioned question is negative in case $\operatorname{int} \arg\min_f(X)$ is allowed to be empty. Indeed, let X and $\mathfrak{N}$ be as in the last example. Let $S := \{(x_1; 0) : |x_1| \leqslant 0.1\}$ and let $g(x_1, x_2) = d((x_1, x_2); S)$. Clearly $g \in \Lambda(X)$ and $\operatorname{int} \arg\min_g(X) = \operatorname{int} S = \varnothing$; but for each C in $\mathfrak{N}$, $\arg\min_g(C) = S$. We intend to show that if we exclude these possibilities, then the answer to the above mentioned question is affirmative.

LEMMA 4.4. *Let X be in (R_f) and be separable. If $f \in \Lambda(X)$, then the set*

$$\Omega \equiv \{C \in CC(X) : C \cap \arg\min_f(X) = \varnothing\}$$

(resp. the set $\Omega \equiv \{C \in CC(X) : C \cap \operatorname{int} \arg\min_f(X) = \varnothing\}$*)*

equipped with the relative topology is a completely metrizable subspace of $(CC(X), \tau_M)$.

Proof. Clearly $\Omega = (\arg\min_f(X)^c)^+$ (resp. $\Omega = (\operatorname{int} \arg\min_f(X)^-)^c$). Since $\arg\min_f(X)$ is w-compact (resp. $\operatorname{int} \arg\min_f(X)$ is open), Ω is an open (resp. closed) subspace of the completely metrizable space $(CC(X), \tau_M)$. It follows from the theorem of Alexandroff [27, p. 179] (resp. follows trivially) that Ω equipped with the relative topology is completely metrizable. ∎

THEOREM 4.5. *Let X be in (R_f) and be separable. Let $CC(X)$ be equipped with the topology τ_M. Assume either*

(1) *$f \in \Lambda(X)$ and $\Omega \equiv \{C \in CC(X) : C \cap \arg\min_f(X) = \varnothing\}$ or*

(2) *$f \in \Lambda(X)$ with $\operatorname{int} \arg\min_f(X) \neq \varnothing$ and $\Omega \equiv \{C \in CC(X) : C \cap \operatorname{int} \arg\min_f(X) = \varnothing\}$.*

If Ω is equipped with the relative topology, then there exists a dense and G_δ subset Ω_0 of Ω such that for each C in Ω_0, $\arg\min_f(C)$ is a singleton.

Proof. This is an obvious modification of the proof of Theorem 4.3 of [4]. We employ Corollary 4.2 and observe that the restriction of the multifunction $\arg\min_f$ to Ω is w-usco. By a continuity theorem due to Christensen [7] there is a dense and G_δ subset Ω_0 of Ω such that this restriction is almost lower semicontinuous on Ω_0. Let $C_0 \in \Omega_0$ and $\alpha = v_f(C_0)$. In case (1) we have $\alpha > v_f(X)$ and the convex set $\text{sub}(f; \alpha)$ has nonvoid interior $\{x \in X: f(x) < \alpha\}$ which has void intersection with C_0. In case (2) if $\alpha > v_f(X)$, we again conclude as above and if $\alpha = v_f(X)$, then $\text{sub}(f; \alpha) = \arg\min_f(X)$ which has nonvoid interior $\text{int} \arg\min_f(X)$ and $\text{int sub}(f; \alpha) \cap C_0 = \varnothing$ by hypothesis. In either case we can use the separation therem and the proof of $\arg\min_f(C_0)$ being singleton can be completed on the same lines as the proof of Theorem 4.3 of [4]. The details are therefore omitted. ∎

A part of the next corollary is already observed in [4].

COROLLARY 4.6. *Let X be in (R_f) and be separable. If $CC(X)$ is equipped with the topology τ_M, then for each nonempty bounded subset F of X (resp. nonempty bounded subset F of X such that* $\text{int Cent}(F; X) \neq \varnothing$), $\text{Cent}(F; C)$ *is a singleton for most $C \in CC(X)$ for which $C \cap \text{Cent}(F; X) = \varnothing$ (resp. $C \cap \text{int Cent}(F; X) = \varnothing$).*

5. APPROXIMATE MINIMAL METRIC SELECTIONS

Throughout this section X will be a Banach space. Let $C \in CC(X)$. A continuous function $f: X \to C$ such that $f(x) \in P(x; C)$ for each x in X is called a *metric selection* for C. It is an easy consequence of the Michael selection theorem [17] that if $P(\cdot; C)$ is l.s.c., then there exists a metric selection for C. By simple examples it is known (cf., e.g., [9]) that the reverse implication is false. For $\varepsilon \geqslant 0$, let $P^\varepsilon(\cdot; C): X \rightrightarrows C$ denote the multifunction with values $P^\varepsilon(x; C) = \varepsilon - P(\theta; P(x; C))$.

DEFINITION. Given $\varepsilon > 0$, a continuous function $f: X \to C$ is called an *ε-approximate minimal metric selection* (resp. a *minimal metric selection*) for C if $f(x) \in P^\varepsilon(x; C)$ (resp. $f(x) \in P^0(x; C)$) for all $x \in X$.

The following theorem characterizes lower semicontinuity of $P(\cdot, C)$.

THEOREM 5.1. *Let C be a closed linear subspace of X. Consider the following statements.*

(1) *For every* $\varepsilon > 0$, *there exists an* ε-*approximate minimal metric selection for* C;

(2) $P(\cdot; C)$ *is l.s.c.*;

(3) *there exists a metric selection* f *with nulleigenschaft*: $f(x) = \theta$ *whenever* $\theta \in P(x; C)$.

Then, we have $(1) \Rightarrow (2) \Leftrightarrow (3)$. *Moreover, in case* $X \in (R_f)$ *we have* $(2) \Rightarrow (1)$ *and all three statements are equivalent.*

Proof. $(1) \Rightarrow (2)$. Let $x_0 \in X$ and V be any open subset of X such that $P(x_0; C) \cap V \neq \emptyset$. Choose $y_0 \in P(x_0; C) \cap V$, then there exists $\varepsilon > 0$ such that $B_\varepsilon(y_0) \subset V$. Since $\theta \in P(x_0 - y_0; C) \cap V - y_0$, by assumption, there exists a metric selection f_ε such that $f_\varepsilon(x_0 - y_0) \in B_\varepsilon(\theta)$. Choose a neighborhood W_0 of $z_0 = x_0 - y_0$ such that $f_\varepsilon(z) \in B_\varepsilon(\theta)$ for every $z \in W_0$. Set $W = W_0 + y_0$, which is a neighborhood of x_0. Let $x \in W$, then $x = z + y_0$ for some $z \in W_0$. We have

$$f_\varepsilon(z) + y_0 \in B_\varepsilon(\theta) + y_0 = B_\varepsilon(y_0) \subset V,$$

and

$$f_\varepsilon(z) + y_0 \in P(z; C) + y_0 = P(x; C).$$

Thus $f_\varepsilon(z) + y_0 \in P(x; C) \cap V$, which implies that $P(x; C) \cap V \neq \emptyset$ for every $x \in W$. Since x_0 is arbitrary, this proves that $P(\cdot; C)$ is l.s.c.

$(2) \Leftrightarrow (3)$. This is proved in [13].

$(2) \Rightarrow (1)$, under the assumption $X \in (R_f)$. Let $P(\cdot; C)$ be l.s.c. For every $\varepsilon > 0$, we claim that $P^\varepsilon(\cdot; C)$ is l.s.c.; hence by the Michael selection theorem [17], ε-approximate minimal metric selection exists. Let $x_0 \in X$ and a sequence $\langle x_n \rangle$ in X be such that $x_n \to x_0$. By lower semicontinuity of $P(\cdot; C)$, it is easy to see that $P(x_n; C) \to^M P(x_0; C)$. Hence by Theorem 3.4, $P^\varepsilon(x_n; C) \to^M P^\varepsilon(x_0; C)$ for every $\varepsilon > 0$. This establishes that $P^\varepsilon(\cdot; C)$ is l.s.c. for every $\varepsilon > 0$. ∎

Remarks 5.2. (1) When $X \in (R_f)$ the proof of $(2) \Rightarrow (1)$ in Theorem 5.1 can be given in an alternative manner as follows. Since $P(\cdot; C)$ is l.s.c., we have $P(x_n; C) \to^M P(x_0; C)$ whenever $x_n \to x_0$. Hence by Theorem 3.4, $d(\theta, P(x_n; C)) \to d(\theta, P(x_0; C))$. This implies that the map $x \to d(\theta, P(x; C)) + \varepsilon$ is continuous for each $\varepsilon \geqslant 0$. Lemma 7.1 in [17] yields the existence of an ε-approximate minimal metric selection.

(2) The proof of $(1) \Rightarrow (2)$ in Theorem 5.1 is patterned on the same lines as in the proof of Proposition 1 in [13].

Brown [6] has proved that $P(\cdot; C)$ is l.s.c. for each finite dimensional subspace of $X \Leftrightarrow X \in (P)$. Following Brown, Wegmann [26] showed that if

$X \in (P)$ then $P(\cdot; C)$ is l.s.c. for every approximatively compact convex set C with property (P): for every $x \in C$, $y \in X$ with $x + y \in C$, there exist positive constants ε, δ such that $z + \varepsilon y \in C$ holds for every $z \in C$ with $\|x - z\| < \delta$.

In this direction, we have the following theorem characterizing lower semicontinuity of $P(\cdot; C)$ in case C is not necessarily a linear subspace.

THEOREM 5.3. *Let $X \in (P)$ and C in $CC(X)$ satisfy property (P). Then $P(\cdot; C)$ is l.s.c. if and only if there exists a metric selection f for C.*

Proof. Suppose there exists a metric selection f and assume $P(\cdot; C)$ is not l.s.c. at x_0. Then there exists a sequence $\langle x_n \rangle$ converging to x_0, $y_0 \in P(x_0; C)$, $y_0 \neq f(x_0)$, and $\delta > 0$ such that

$$P(x_n; C) \cap B_\delta(y_0) = \varnothing \qquad \text{for each } n. \tag{1}$$

Since $X \in (P)$ and C satisfies property (P), there exist positive numbers $\varepsilon_1, \varepsilon_2$ such that $\|x_n - (f(x_n) + \varepsilon_1(y_0 - f(x_0))\| \leqslant \|x_n - f(x_n)\|$ and $f(x_n) + \varepsilon_2(y_0 - f(x_0)) \in C$ hold eventually. Since C is convex, it can be easily concluded that there exists $\varepsilon \in (0, 1)$ such that both $\|x_n - (f(x_n) + \varepsilon(y_0 - f(x_0))\| \leqslant \|x_n - f(x_n)\|$ and $f(x_n) + \varepsilon(y_0 - f(x_0)) \in C$ hold eventually. Thus we have a sequence $\langle f(x_n) + \varepsilon(y_0 - f(x_0)) \rangle$ such that $f(x_n) + \varepsilon(y_0 - f(x_0)) \in P(x_n; C)$ converging to $f(x_0) + \varepsilon(y_0 - f(x_0)) \in P(x_0; C)$. Replacing $f(x_n)$ by $f(x_n) + \varepsilon(y_0 - f(x_0))$ and $f(x_0)$ by $f(x_0) + \varepsilon(y_0 - f(x_0))$ and repeating exactly the same arguments as above leads to a contradiction of (1). Hence $P(\cdot; C)$ is l.s.c. The "only if" part follows trivially from the Michael selection theorem. ∎

In the following theorem we examine the continuity of $P^0(\cdot; C)$.

THEOREM 5.4. *Let $X \in (P)$ and let C be an approximatively compact convex subset satisfying property (P). Then $P^0(\cdot; C)$ is continuous.*

Proof. Let $x_0 \in X$ and let $\langle x_n \rangle$ be any sequence converging to x_0. Let $\langle y_n \rangle$ be any sequence such that $y_n \in P^0(x_n; C)$. Since $P(\cdot; C)$ is usco [25, Proposition 2.9], the sequence $\langle y_n \rangle$ has a subsequence $\langle y_{n(k)} \rangle$ converging to z_0 in $P(x_0; C)$.

To prove upper semicontinuity of $P^0(\cdot; C)$, it is sufficient to show that $z_0 \in P^0(x_0; C)$, that is, $\|z_0\| = d(\theta, P(x_0; C))$. Assume the contrary. Then there is an element z_1 in $P(x_0; C)$ such that $\|z_1\| < \|z_0\|$. Since $X \in (P)$ and C is convex with property (P), exactly as in the proof of the last theorem we can show that there exists $\varepsilon \in (0, 1)$ such that

$$\|y_{n(k)} + \varepsilon(z_1 - z_0)\| \leqslant \|y_{n(k)}\|,$$

$$y_{n(k)} + \varepsilon(z_1 - z_0) \in C$$

and

$$\|x_{n(k)} - (y_{n(k)} + \varepsilon(z_1 - z_0))\| \leqslant \|x_{n(k)} - y_{n(k)}\|$$

hold eventually. Thus we have a sequence $y_{n(k)} + \varepsilon(z_1 - z_0) \in P^0(x_{n(k)}; C)$ converging to $z_0 + \varepsilon(z_1 - z_0)$ such that $\|z_0 + \varepsilon(z_1 - z_0)\| = \|z_0\|$. Replacing $y_{n(k)}$ by $y_{n(k)} + \varepsilon(z_1 - z_0)$ and z_0 by $z_0 + \varepsilon(z_1 - z_0)$ and repeating the above arguments contradict the assumption $\|z_1\| < \|z_0\|$. Thus $z_0 \in P^0(x_0; C)$.

The proof of lower semicontinuity of $P^0(\cdot; C)$ follows exactly from the same contradiction argument as in the previous theorem by taking $y_{n(k)}$ in place of $f(x_{n(k)})$ and z_0 in the place of $f(x_0)$. ∎

Remark 5.5. By the Michael selection theorem, Theorem 5.4 gives existence of (minimal) metric selection, and hence in conjunction with Theorem 5.3, yields Theorem 5.5 of [26].

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